MULTICOLORING AND WAVELENGTH ASSIGNMENT IN WDM ALL-OPTICAL NETWORKS

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ABSTRACT

This paper deals with the wavelength assignment problem in WDM all-optical networks. This problem can be abstracted as a multicoloring problem on a weighted conflict graph. The main focus of this work is to obtain the relationship between the wavelength assignment and the multicoloring problems. We first, establish a new result regarding multicoloring problem in which we provide a tight bound for any graph. Afterwards, considering the off-line wavelength assignment problem in all-optical networks particularly for certain routing patterns, we show that coloring all paths of this routing is the same as the multicoloring problem of the associated conflict graph.

Keywords: WDM all-optical network; Routing; Graph theory, Graph multicoloring; Approximation algorithm.

1 INTRODUCTION

In *all-optical networks* that use the *wavelength division multiplexing* (WDM) technology, each optical fiber can carry several signals, each on a different wavelength. Thus, a typical problem (known as the *wavelength routing problem*) is to accept connection requests in the network. That is, for each connection request, to find a path in the network and to allocate it a wavelength such that no two paths with the same wavelength share a link. In order to make an optimal use of the available bandwidth, it is important to control two parameters: the maximum number of paths that cross a link (the *load*) and the total *number of wavelengths* used. In this paper we will consider only the *wavelength assignment problem* (also known as the path coloring problem) which consists, for a set of paths, in assignating to each path a wavelength (color). Paths of the same color must be edge-disjoint. Our objective is to minimize the number of colors used.

An optical network is often modelized by a directed *graph* where the vertices represent the optical switches and the edges represent the optical fibers. Let G=(V,E) be a directed graph (digraph) with vertex set V and (directed) edge set E. We will consider only symmetric digraphs (i.e. $(x,y) \in E \Rightarrow (y,x) \in E$), but they will be drawn as undirected graphs. An *instance* I is a collection of requests (i.e. pairs of nodes that request a connection in the network):
$I=\{(x_i ,y_i)|\ x_i ,y_i \in V(G)\}$. Note that a request can appear more than once in I. A *routing* $\boldsymbol{R}$ is a multiset of dipaths that realize I, i-e. to each connection of I corresponds a dipath in $\boldsymbol{R}$.
The *load* $L(G,\boldsymbol{R},e)$ of an edge e for a routing $\boldsymbol{R}$ is the number of dipaths that cross e. The load $L(G,\boldsymbol{R})$ of a routing $\boldsymbol{R}$ is the maximum of the load of any edge: $L(G,\boldsymbol{R})= \max_{e\in E} L(G,\boldsymbol{R},e)$.
The *wavelength number* $\lambda(G,\boldsymbol{R})$ for a routing $\boldsymbol{R}$ is the minimum number of wavelengths needed by the dipaths of $\boldsymbol{R}$ in such a way that no two paths sharing an edge get the same wavelength. The wavelength number is also the *weighted chromatic number* of the *weighted conflict graph*, where each vertex corresponds to a path of $\boldsymbol{R}$ such that the weight of each vertex is the multiplicity of the associated path and where two vertices are adjacent if the corresponding paths share a common edge in G. Observe that, for any routing, the load is a lower bound on the wavelength number. A coloring algorithm $\boldsymbol{A}$ is said to be a *p-approximation* if for any routing $\boldsymbol{R}$, the number of wavelength $\lambda_{\boldsymbol{A}}(G,\boldsymbol{R})$ needed by the algorithm is at most at a factor p from the optimal, that is, $\lambda_{\boldsymbol{A}}(G,\boldsymbol{R}) \leq p.\ \lambda(G,\boldsymbol{R})$.
The wavelength routing problem has been extensively studied and proved to be a difficult problem, even when restricted to simple network topologies (for instance, it is NP-complete for trees and for cycles [2]). Much work has been done on approximation or exact algorithms to compute the wavelength number for particular networks or for particular instances [1,10].
In mesh networks, for the wavelength routing problem, the best known algorithm [9] has an approximation ratio poly(ln(lnN)) for a square undirected grid of order N, where poly is a polynom. For the coloring problem on meshes, [8] showed that it is NP-complete and NoAPX. Moreover, it is proved in [8] that this problem remains NP-complete even if restricted to line-column or column-line paths.

The paper is organized as follows. The next Section focuses on the multicoloring problem of a general graph. In Section 3, we show that wavelength assignment problem on some graphs is the same as the multicoloring problem of the associated conflict graph. Section 4 give methods to color class of routing, called (α,β) LC-routing, based on the vertices multicoloring of weighted conflict graph. In Section 5 and Section 6, we extend the ressult of Section 4 to line-column routings in toroidal meshes and to (line-column, column-line) routings in meshes.

2 MULTICOLORING

Given a graph G=(V,E), we define its *chromatic number* to be the smallest number of colors needed such that every vertex is assigned a color and no two vertices connected by an edge receive the same color. The *independence number* of G is defined to be the largest subset of vertices such that no two of them are connected by an edge in G. We use χ(G) and α(G) to denote the chromatic number and the independence number of G.

A *weighted graph* of G is a pair G_ω =(G, ω) where ω is a weight function that assigns non-negative integer ω(v) to each vertex v of G, ω(v) is called the *weight* of v. A vertex multicoloring of the weighted graph G_ω consists of a set of colors ***C*** and a function Ψ that assigns to each v ∈ V a subset of colorsΨ(v) ⊂ C such that:

i) ∀ v ∈V, |Ψ(v)|= ω(v), i-e. the vertex v gets ω(v) distinct colors.

ii) If (u,v) ∈ E then Ψ(u) ∩ Ψ(v)=∅, i-e. two adjacent vertices get disjoint sets of colors.

The *weighted chromatic number*, denoted $\chi_p(G_\omega)$, of G_ω is the minimum number of colors needed to color all vertices of G_ω so that conditions **i)** and **ii)** above are satisfied.

The multicoloring problem (also known as weighted coloring [3] or ω-coloring [7] is NP-hard in general. Hence, it would be interesting to find algorithms that approximate the weighted chromatic number. A vertex subset K of G_ω is called a clique if every pair of vertices in K are adjacent. The weight of any clique in G_ω is defined as the sum of the weights of the vertices forming that clique. The *weighted clique number* of G_ω, denoted $W(G_\omega)$ (for short, we use W), is defined to be the maximum over the weights of all cliques in G_ω. Then the following theorem states the relationship between these parameters.

Theorem 1

For any graph G=(V,E) such that |V| ≥ 2 and |E| ≥ 2, we have

$$\max\left(W, \frac{\sum_{v\in V}\omega(v)}{\alpha(G)}\right) \le \chi_p(G_\omega) \le \frac{W}{2}.\chi(G)$$

Proof

Let G=(V,E) be a general graph and G_ω its weighted graph. First, the following inequality is immediate from the above definitions.

$$\max\left(W, \frac{\sum_{v\in V}\omega(v)}{\alpha(G)}\right) \le \chi_p(G_\omega)$$

Hence, it remains to show that

$$\chi_p(G_\omega) \le \frac{W}{2}.\chi(G)$$

To prove this, we describe an algorithm that proceeds in two steps.

Step 1:

First, we take a proper coloring of G with χ(G) colors that we call the basic coloring of G_ω. Next, we assume that G_ω is connected, since disconnected components of G_ω can be independently colored without any color conflict. Let $l = \frac{W}{2}$. Every vertex v is assigned the first hues interval $[1, l\,]$ of the base color of the vertex. Note that a vertex v is called *heavy* if ω(v) > l and is called *light* if ω(v) ≤ l. Hence, only the heavy vertices remain to be completely colored after this step and the light vertices are completely colored and are deleted from G_ω. Let $H_{\omega'}$ denote the remaining graph obtained after this process where the new weight of each vertex v∈ $H_{\omega'}$ is ω'(v) = ω(v)- l. We observe that $H_{\omega'}$ contains only isolated vertices, because if we suppose that there exist two heavy adjacent vertices v_1 and v_2 in $H_{\omega'}$ then, we get $2l < \omega(v_1) + \omega(v_2) \le W = 2l$, a contradiction.

Step 2:

Consider v_j a vertex of $H_{\omega'}$ having j as base color in G_ω. Then, according to Step 1, v_j is a heavy vertex in G_ω and all its neighbors must be light vertices in G_ω. Let N_i be the set of neighbors vertices of v_j having i≠j as base color in G_ω. Let $v_i \in N_i$ such that $\omega(v_i) = \max_{v\in N_i}\omega(v)$.

Further, we have $\omega(v_j) = l + \varepsilon_j$ and $\omega(v_i) = l - \varepsilon_i$ where $0 \le \varepsilon_j, \varepsilon_i \le l$. As $\omega(v_j) + \omega(v_i) \le W = 2l$, we get $\varepsilon_j \le \varepsilon_i$. Then v_j can borrow from ε_i colors available in the $[1, l\,]$ hues of base color i for coloring the remaining weight on v_j.

Consequently, for multicoloring all vertices of G_ω, we use at least χ(G). Thus,

$$\chi_p(G_\omega) \le \frac{W}{2}.\chi(G)$$

3 WAVELENGTH ASSIGNMENT AND GRAPH MULTICOLORING

The wavelength assignment problem (WAP) in a WDM all-optical network is generally modelized as a graph theoretical coloring. In this paper, we describe a new approach where the WAP is abstracted as a multicoloring problem on a weighted conflict graph.

Given a digraph G and a routing $\boldsymbol{R}$ represented by a set of pairs (P,M(P)) where P is a dipath on G and M(P) its multiplicity, *the weighted conflict graph* $G_{cp}=(V_{cp},E_{cp})$ of $\boldsymbol{R}$ is a weighted graph where each vertex v_p corresponds to a path P of $\boldsymbol{R}$, two vertices are adjacent if the corresponding paths share a common edge in G and where the weight $\omega(v_p)$ of vertex v_p is the multiplicity M(P) of the corresponding path P in $\boldsymbol{R}$ (See FIGURE 1).

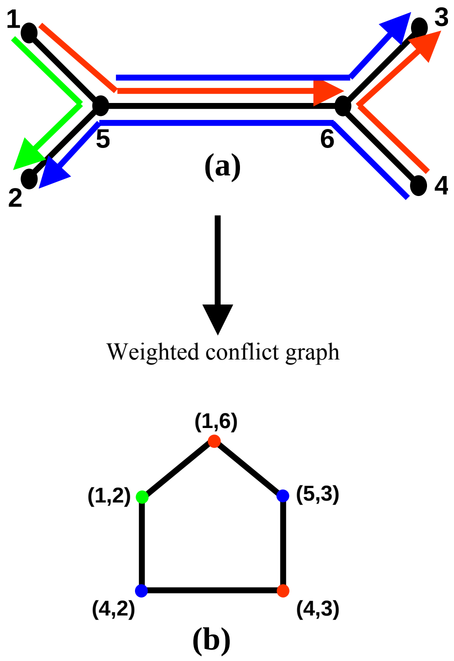

FIGURE 1: (a): A SET OF DIPATHS ON A GRAPH. (b): THE CORRESPONDING CONFLICT GRAPH.

Proposition 1

Given a graph G, for any routing $\boldsymbol{R}$ in G, we have

1. $\lambda(G, \boldsymbol{R}) = \chi_\omega(G_{cp})$,
2. $L(G, \boldsymbol{R}) \leq W(G_{cp})$.

Proof

Let G be a graph and let $\boldsymbol{R}$ be a routing in G.

1. Is immediate from the above definitions.
2. There exists $e \in E$ such that $L(G,\boldsymbol{R})=L(G,\boldsymbol{R},e)$. Let $K=\{(P,M(P)) \in \boldsymbol{R} | e \in P\}$ be a subset of $\boldsymbol{R}$. As $L(G,\boldsymbol{R})$ is the number of paths that cross e, we get $L(G,\boldsymbol{R}) = \sum_{P \in K} M(P)$. Consider $Q=\{(v_p, \omega(v_p)) \in V_{cp} | P \in K\}$ be a subset of V_{cp} where each vertex v_P corresponds to a path P of K and $\omega(v_p)=M(P)$. We observe that Q is a clique of G_{cp} because for any pair of vertices $(v_P,v_{P'}) \in Q$, the corresponding paths $(P,P') \in K$ are in conflict ($e \in P \cap P'$). Further, we have
$$L(G,\boldsymbol{R}) = \sum_{P \in K} M(P) = \sum_{v_P \in Q} \omega(v_p)$$

Thus,
$$L(G, \boldsymbol{R}) \leq W$$

In general, determining the wavelength number is an intractable problem. For this reason, in what follows, we study various usual graph families on which we can determine the relationship between the wavelength assignment and the multicoloring problems.

3.1 PATH AND CYCLE GRAPH

By $\boldsymbol{P}_n$ (resp. $\boldsymbol{C}_n$) we denote the path graph (resp. cycle graph) of order n, with vertex set $V_n=\{0,1,...,n-1\}$.
For a graph G, we write G^p for the *pth* power of G i-e. the graph on the same vertex set than G and with edges linking vertices at distance at most p in G.

Theorem 2

For any routing $\boldsymbol{R}$ on the path graph $\boldsymbol{P}_n$ with load L, we have L = W.

Proof

Let $G_{cp}(\boldsymbol{R})$ be the weighted conflict graph of $\boldsymbol{R}$, we have $L(\boldsymbol{P}_n,\boldsymbol{R}) = \lambda(\boldsymbol{P}_n,\boldsymbol{R})$ and $\lambda(\boldsymbol{P}_n,\boldsymbol{R}) = \chi_\omega(G_{cp}) \geq W$. Then, Proposition 1 gives us L = W.

Proposition 2

There exists a routing $\boldsymbol{R}$ of load L on the cycle $\boldsymbol{C}_n$ such that L < W.

Proof

Let $\boldsymbol{R}=\{((0,\left\lceil \frac{n}{2} \right\rceil),k),((2,n-1),k),((n-2,1),k)\}$ be a routing in $\boldsymbol{C}_n$ where k is the multiplicity of each path. We observe that $L(\boldsymbol{C}_n, \boldsymbol{R}) = 2k$ but the conflict graph of $\boldsymbol{R}$ is a triangle so that each vertex weight are k. Hence, the weighted clique number of this triangle is W = 3k. Consequently, $L(\boldsymbol{C}_n, \boldsymbol{R}) < W$ (See FIGURE 2).

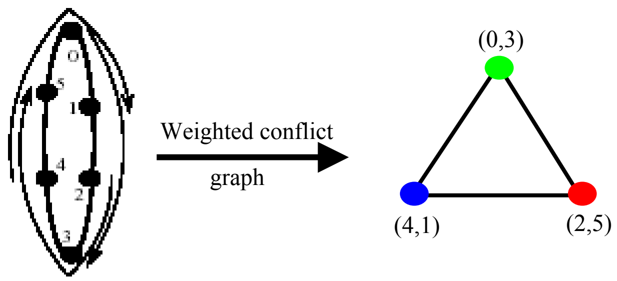

FIGURE 2: LEFT: A 3-PATH SET ON CYCLE $\boldsymbol{C}_6$. RIGHT: THE CORRESPONDING WEIGHTED GRAPH CONFLICT. WE HAVE L=2 AND ω=3.

Now, consider a particular routing $\boldsymbol{R}_l$ in $\boldsymbol{C}_n$ such that each path has *l* as length.

Proposition 3

The path coloring problem of $\boldsymbol{R}_l$ in $\boldsymbol{C}_n$ is the same as the multicoloring problem of the graph $\boldsymbol{C}_n^{\,l}$.

Proof

Denote by P_i a path of $\boldsymbol{R_l}$ which starts at vertex i. Observe that for every vertex $i \in V_n$, the path P_i is in conflict with all paths P_j with $i-l+1 \pmod n \le j \le i+l-1 \pmod n$. Thus, it is easy to verify that the associated conflict graph is the $(l-1)th$ power cycle $\boldsymbol{C}_n^{l-1}$.

3.2 MESH, TOROIDAL MESH

The coloring problem on meshes is NP-complete and NoAPX as stated in the Introduction. Moreover, it is proved that this problem remains NP-complete even if restricted to line-column or column-line paths. For this reason, here we study only the LC-routing on the mesh and on the toroidal mesh.

Definition 1

The 2-dimensional symmetric directed mesh $M_{m,n} = (V(M_{m,n}), E(M_{m,n}))$ has vertex set $V(M_{m,n}) = \{(i,j) \mid 1 \le i \le m;\ 1 \le j \le n\}$ and edge set $E(M_{m,n}) = E_R \cup E_L \cup E_U \cup E_D$ where:

- $E_R = \{((i,j),(i,j+1)),\ 1 \le i \le m;\ 1 \le j \le n-1\}$ right edge (R-edge for short) set.
- $E_L = \{((i,j),(i,j-1)),\ 1 \le i \le m;\ 2 \le j \le n\}$ left edge (L-edge) set.
- $E_U = \{((i,j),(i-1,j)),\ 2 \le i \le m;\ 1 \le j \le n\}$ up edge (U-edge) set.
- $E_D = \{((i,j),(i+1,j)),\ 1 \le i \le m-1;\ 1 \le j \le n\}$ down edge (D-edge) set.

The *toroidal mesh* is a mesh on the torus, i.e. the 2-dimensional toroidal mesh $TM_{m,n}$ has vertex set $V = V(M_{m,n})$ and edge set $E = E(M_{m,n}) \cup E_T$, where E_T consists of

- R-edges: $\{((i,n),(i,1)),\ 1 \le i \le m\}$,
- L-edges: $\{((i,1),(i,n)),\ 1 \le i \le m\}$,
- D-edges: $\{((m,j),(1,j)),\ 1 \le j \le n\}$,
- U-edges: $\{((1,j),(m,j)),\ 1 \le j \le n\}$.

Definition 2

Let $X \in \{R,L,U,D\}$, $Y \in \{R,L\}$ and $Z \in \{U,D\}$. Let G be the mesh $M_{m,n}$ or the toroidal mesh $TM_{m,n}$.

- An X-path in G is a path using only X-edges.
- An YZ-path in G is a path formed by a concatenation of a Y-path with a Z-path.
- A Line-Column routing or LC-routing in G is a routing which only consists of X-paths and YZ-paths.
- An XY-routing is a line-column routing which contains only XY-paths.
- An (LC,CL)-routing in G is a routing which contains only YZ-paths and ZY-paths.
- An (α,β) LC-routing is a LC-routing where each path consists of exactly α horizontal edges and β vertical edges.
- An $(\alpha,*)$ LC-routing is a LC-routing where each path consists of exactly α horizontal edges.
- The XY-path coloring problem is the path coloring problem restricted to XY-routings.

Notice that a LC-routing is composed of 8 types of paths, namely: R, L, U, D, RD, RU, LD, LU. But to simplify, in the following, we will consider a R-path or a D-path to be a RD-path and a L-path or a U-path to be an LU-path. Thus we only deal with YZ-paths.

Proposition 4

If $\boldsymbol{R}$ is a LC-routing with load L on the mesh $M_{m,n}$ then L=W.

Proof

Let $\boldsymbol{R}$ be a LC-routing on $M_{m,n}$ with load L and let $P \in \boldsymbol{R}$. We observe that P is formed by concatenation of a horizontal path P^H and a vertical path P^V. Then we can write $P = (P^H, P^V)$ and $\boldsymbol{R} = \boldsymbol{R^H} \times \boldsymbol{R}^V$ where $\boldsymbol{R^H}$ (resp. $\boldsymbol{R^V}$) is a set of horizontal paths (resp. vertical paths). Further, $\boldsymbol{R^H}$ (resp. $\boldsymbol{R^V}$) may be partitioned into m sets of horizontal paths $\boldsymbol{R^H_i}$ on the path graph $\boldsymbol{P}_n^i$ with $1 \le i \le m$ (resp. n sets of vertical paths $\boldsymbol{R^V_j}$ on the path graph $\boldsymbol{P}_m^j$ with $1 \le j \le n$. It is easy to verify that $L = \max(L(\boldsymbol{R^H}), L(\boldsymbol{R^V}))$ where $L(\boldsymbol{R^H}) = \max\limits_{1 \le i \le m} (L(\boldsymbol{R^H_i}, \boldsymbol{P}_n^i))$ and $L(\boldsymbol{R^V}) = \max\limits_{1 \le j \le n} (L(\boldsymbol{R^V_j}, \boldsymbol{P}_m^j))$.

Now, suppose that there exist three paths P_1, P_2, P_3 of different type which are in conflict between them. Since, if P_1 is of type RD then P_2 is of type RU or of type LD. If P_2 is of type RU then P_3 must be of type LU a contradiction because P_1 is in conflict with P_3. Consequently, if three paths are in conflict between them then at least two paths among them are of the same type and there exists an edge crossed by these paths. This means that, if K is a clique of weighted conflict graph of $\boldsymbol{R}$ then all paths corresponding to vertices of K belong to at most two different types of paths. In other term, if R_K is the set of paths corresponding to vertices of K then $\boldsymbol{R_K} = \{$XY-paths|XZ-paths$\}$ where $X \in \{R,L\}$ and $Y \neq Z \in \{D,U\}$ or $\boldsymbol{R_K} = \{$XY-paths|ZY-paths$\}$ where $X \neq Z \in \{R,L\}$ and $Y \in \{D,U\}$. Moreover, as $\boldsymbol{R}_K = \boldsymbol{R^H_i} \times \boldsymbol{R^V_j}$ with $\boldsymbol{R^H_i}$ is a routing on the path graph $\boldsymbol{P}_n^i$ and $\boldsymbol{R^V_j}$ is a routing on the path graph $\boldsymbol{P}_m^j$, Theorem 2 gives $\omega(K) = \max(L(\boldsymbol{R^H_i}, \boldsymbol{P}_n^i), L(\boldsymbol{R^V_j}, \boldsymbol{P}_m^j))$. Thus,

$$W = \max_{i,j} (L(\boldsymbol{R^H_i}, \boldsymbol{P}_n^i), L(\boldsymbol{R^V_j}, \boldsymbol{P}_m^j)) = \max(L(\boldsymbol{R^H}), L(\boldsymbol{R^V})) = L$$

4 COLORING (α,β) LC-ROUTINGS

Proposition 5

The (α,β) XY-path coloring problem in $M_{m,n}$ is the same as the multicoloring problem of the graph $P_{m-\beta}^{\alpha-1} \square P_{n-\alpha}^{\beta-1}$.

Proof

Without loss of generality, we consider $\boldsymbol{R}$ a (α,β) RD-routing in $M_{m,n}$. We denote by P_{ij} the path in $\boldsymbol{R}$ which starts at vertex (i,j), i-e. $P_{ij} = (i,j)(i,j+1)...(i,j+\alpha)(i+1,j+\alpha)...(i+\beta,j+\alpha)$ where $1 \le i \le m-\beta$ and $1 \le j \le n-\alpha$. Let $G_{cp} = (V_{cp}, E_{cp})$ be the corresponding weighted conflict graph of ***R*** such that for each path $P_{ij} \in R$, we associate the vertex $(i,j) \in V_{cp}$. Then $V_{cp}=\{1...m-\beta\}\times\{1...n-\alpha\}$. Note that the weight of (i,j) is $\omega((i,j))=M(P_{ij})$ where $M(P_{ij})$ is the multiplicity of the path P_{ij}. Further, for any $1 \le k \le \beta -1$ and $1 \le l \le \alpha - 1$ we have $((i,j),(i+k,j)) \in E_{cp}$ and $((i,j),(i,j+l)) \in E_{cp}$ because the path P_{ij} is in conflict with $P_{(i+k)j}$ and with $P_{i(j+l)}$. Thus it is easy to observe that $G_{cp}= P_{m-\beta}^{\alpha-1} \square P_{n-\alpha}^{\beta-1}$.

In the following, we give some results concerning particular values of α and β. Note that, for symmetrical reason, α and β play the same role.

4.1 α=1 AND β IS ARBITRARY

Proposition 6

There exist (1,*) LC-routings ***R*** on the mesh $M_{m,n}$ that require $\lceil \frac{5}{4} L(M_{m,n}, \boldsymbol{R}) \rceil$ colors.

Proof

Let ***R*** be the (1,*) LC-routing of load L=2k obtained by taking k copies of each path given in the FIGURE 3.

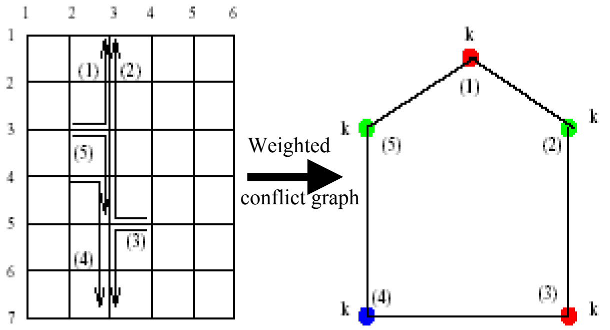

FIGURE 3: THE (1,*) LC-ROUTING ON THE MESH WITH L=2k AND $\lambda=\lceil \frac{5}{4} L \rceil$.

It is easy to see that the corresponding weighted conflict graph is a cycle on five vertices. Then, according to [7], the path coloring of ***R*** requires $\lceil \frac{5}{4} L(M_{m,n}, \boldsymbol{R}) \rceil$ colors. Thus, the proposition is proved.

Proposition 7

There exists a polynomial time algorithm that colors any (1,*) LC-routing ***R*** on the mesh $M_{m,n}$ using at most 2 $L(M_{m,n}, \boldsymbol{R})$ colors.

Proof

Consider the RD-paths and LD-paths of ***R***. As a path on column c can not conflict with a path on column c'≠c thus, coloring RD-paths and LD-paths of ***R*** is equivalent to coloring paths on each column separately. And it is straightforward that coloring a set of paths with load L on a linear graph can be done optimally by a polynomial time algorithm using at most L colors. Thus, we need at most $L(M_{m,n}, \boldsymbol{R})$ colors for coloring the RD-paths and LD-paths. Again, for symmetrical reasons, we use a new set of $L(M_{m,n}, \boldsymbol{R})$ colors for coloring the RU-paths and LU-paths of ***R***. Therefore, for coloring all paths of ***R***, we use at most 2 $L(M_{m,n}, \boldsymbol{R})$ colors.

4.2 α=2 AND β=2

Proposition 8

There exist (2,2) LC-routings ***R*** on the mesh $M_{m,n}$ that require $\lceil \frac{7}{6} L(M_{m,n}, \boldsymbol{R}) \rceil$ colors.

Proof

Let $M_{7,6}=(V,E)$ be a mesh with vertex set $V=\{(i,j) | 1 \le i,j \le 7\}$. Let ***R***={((3,1),(1,3));((3,1),(5,3));((4,1),(6,3));((5,1),(3,3)); ((5,2),(7,4));((4,5),(2,3));((4,2),(6,4))} be a (2,2) LC-routing of load L=2k where k is the multiplicity number of each path of ***R*** (See FIGURE 4).

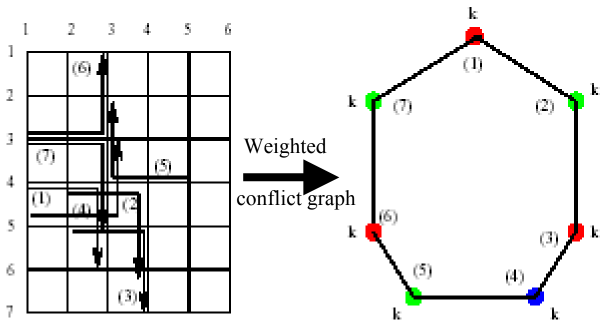

FIGURE 4 : THE L-ROUTING ON THE MESH WITH L=2k AND $\lambda=\lceil \frac{7}{6} L \rceil$.

Moreover, The corresponding weighted conflict graph is an odd simple cycle of order 7. Then, according to [7], the path coloring of ***R*** requires $\lceil \frac{7}{6} L(M_{m,n}, \boldsymbol{R}) \rceil$ colors.

Proposition 9

There exists a polynomial time algorithm that colors any (2,2) LC-routing ***R*** on the mesh $M_{m,n}$ using at most $2L(M_{m,n},\boldsymbol{R})$ colors.

Proof

Consider a sub-routing $\boldsymbol{R}_{RD}$ with load L of ***R*** which contains only the (2,2) RD-paths of R. According to Proposition 5, the (2,2) RD-paths coloring problem in $M_{m,n}$ is the same as the multicoloring problem of $P_{m-2} \square P_{n-2}$. Moreover, it is known that $P_{m-2} \square P_{n-2}$ is a 2-dimension mesh of order (m-2)×(n-2), thus the chromatic number of $P_{m-2} \square P_{n-2}$ is 2. So, according to Theorem 1, we get $\chi_p(P_{m-2} \square P_{n-2}) \le W$. Theorem gives us W = L and as $W \le \chi_p(P_{m-2} \square P_{n-2})$, we obtain $\chi_p(P_{m-2} \square P_{n-2}) = W$.

Therefore, for coloring all paths of $\boldsymbol{R}_{RD}$, we use at most L colors. Again, for symmetrical reason, we can use a new set of L colors for coloring the paths of $\boldsymbol{R}_{LD}$. Consequently, for coloring all paths of $\boldsymbol{R}$, we use at most 2L colors.

4.3 α=3 AND β=2 OR 3

Proposition 10
If α=3 and β =2 or 3 then there exists a polynomial time algorithm that colors any (α,β) LC-routing $\boldsymbol{R}$ on the mesh $M_{m,n}$ using at most $3L(M_{m,n},\boldsymbol{R})$ colors.

Proof
Consider a (α,β) LC-routing $\boldsymbol{R}$ with load L on the mesh $M_{m,n}$ where α = 3 and β = 2 or 3 . For proof, without loss of generality, we take β = 2. First, let $\boldsymbol{R_{RD}}$ be the RD path subset of $\boldsymbol{R}$. Using Proposition 5, the weighted conflict graph corresponding to $\boldsymbol{R_{RD}}$ is $P_{m-3}^{2} \square P_{n-3}^{2}$. Note that $P_{m-3}^{2} \square P_{n-3}^{2}$ = (V,E) with V = {(i,j)|0 ≤ i ≤ m-4; 0 ≤ j ≤ n-4} and E={((i,j),(i± r,j));((i,j),(i,j±r))|(i,j)∈V;1 ≤ r ≤ 2}. Moreover, for each vertex (i,j) of V, we assign it the color (i+j) mod(3). In addition, if (i,j) and (i± r,j) have the same color we obtain i+j = i+j±r mod(3), this gives r=0 mod(3), a contradiction because 1 ≤ r ≤ 2. Hence, (i,j) and (i±r,j) have no color conflicts. Therefore, $P_{m-3}^{2} \square P_{n-3}^{2}$ has a vertex coloring in three colors{0,1,2}. According to Theorem 1 and Theorem 2, we get $\chi_p(P_{m-3}^{2} \square P_{n-3}^{2}) \le \frac{3}{2}$ L. Then, the path coloring of $\boldsymbol{R}$ needs at most $\frac{3}{2}$ L colors. As the LU-paths are not in conflict with the RD-paths, we can use the same colors set to color them. Similarly, for coloring the RU-paths and the LD-paths we use a new class of $\frac{3}{2}$ L colors. So, for coloring all paths of $\boldsymbol{R}$, we can use at most 3L colors.

4.4 α=β IS ARBITRARY

We saw in Proposition 5 that the (α,β) XY-path coloring problem is the same as the multicoloring problem of $P_{m-\beta}^{\alpha-1} \square P_{n-\alpha}^{\beta-1}$. Thus, according to the following theorem 2, we can color all paths of any (α,α) XY-routing of load L using at most 2L colors.

Theorem 3 (KCHIKECH and TOGNI [5])
Let G= $P_{m}^{\alpha} \square P_{n}^{\alpha}$. For any weighted graph G_ω of G, there exists a polynomial time algorithm which multicolors all vertices of G_ω using at most 2W colors.

Theorem 4
There exists a polynomial time algorithm that colors any (α,α) XY-routing $\boldsymbol{R}$ on the mesh $M_{m,n}$ using at most $2L(M_{m,n}, \boldsymbol{R})$ colors.

Proof
Proposition 5 gives us $\lambda(M_{m,n}, \boldsymbol{R})=\chi_p(P_{m-\beta}^{\alpha-1} \square P_{n-\alpha}^{\alpha-1})$. According to Theorem 3 and to Proposition 4 we get $\lambda(M_{m,n}, \boldsymbol{R}) \le 2L(M_{m,n}, \boldsymbol{R})$.

Corollary 1
There exists a polynomial time algorithm that colors any (α,α) LC-routing $\boldsymbol{R}$ on the mesh $M_{m,n}$ using at most $4L(M_{m,n}, \boldsymbol{R})$ colors.

Proof
As $\lambda(M_{m,n}, \boldsymbol{R}) \le \lambda(M_{m,n}, \boldsymbol{R}_{RD}) + \lambda(M_{m,n}, \boldsymbol{R}_{LD})$, we get

$$\lambda(M_{m,n}, \boldsymbol{R}) \le 4L(M_{m,n}, \boldsymbol{R})$$

5 COLORING LC-ROUTINGS ON THE TOROIDAL MESH

Theorem 5 (KCHIKECH and TOGNI [6])
Let I be an instance on the toroidal mesh $TM_{m,n}$. For any LC-routing $\boldsymbol{R}$ realizing I with load $L=L(TM_{m,n}, \boldsymbol{R})$, there exists a LC-routing $\boldsymbol{R}$ ' on $M_{m,n}$ realizing I with load L' ≤ 2L and there exists a polynomial time greedy algorithm which colors all paths of $\boldsymbol{R}$ ' using at most 4L' ≤ $8L(TM_{m,n}, \boldsymbol{R})$) colors.

For case where $\boldsymbol{R}$ contains only (α,β)-paths, we have the following result:

Proposition 11
The (α,β) XY-path coloring problem on the toroidal Mesh $TM_{m,n}$ where X∈{R,L} and Y∈{U,D} is the same as the multicoloring problem of $C_{m}^{\alpha-1} \square C_{n}^{\beta-1}$. Where $C_{m}^{\alpha-1}$ is the (α-1)*th* power of the cycle C_m of order m.

Proof
For this proof, we proceed in the same way as for the proof of Proposition 5.

6 COLORING (LC,CL)-ROUTINGS

Theorem 6
Let G be a mesh or a toroidal mesh. If there exists a polynomial time p-approximation algorithm to color any LC-routing in G then there exists a polynomial time 2p-approximation algorithm to color any (LC,CL)-routing in G.

Proof
Let X∈{R,L} and let Y∈{U,D}. Suppose that one can color XY-paths of R with no more than pL colors. Then, by symmetry, we can color the YX-paths with another set of at most pL colors.

Corollary 2
There exists a polynomial time 8-approximation algorithm that colors any (α,α) (LC,CL)-routing in $M_{m,n}$.

Corollary 3

For any LC-routing $\boldsymbol{R}$ in the toroidal mesh $TM_{m,n}$, there exists a (LC,CL)-routing $\boldsymbol{R}'$ such that there is a polynomial time algorithm that colors all paths $\boldsymbol{R}'$, using at most 16L ($TM_{m,n}$, $\boldsymbol{R}$) colors.

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